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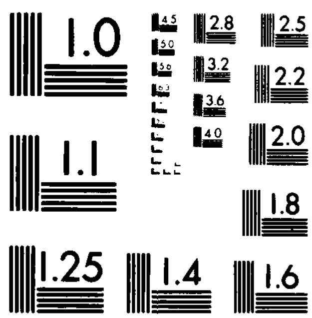

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OPTIMAL SIMPLEX TABLEAU CHARACTERIZATION OF UNIQUE AND BOUNDED SOLUTIONS OF LINEAR PROGRAMS

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ABSTRACT

Uniqueness and boundedness of solutions of linear programs are characterized in terms of an optimal simplex tableau. Let M denote the submatrix in an optimal simplex tableau with columns corresponding to degenerate optimal dual basic variables. A primal optimal solution is unique if and only if there exists a nonvacuous nonnegative linear combination of the rows of M corresponding to degenerate optimal primal basic variables which is positive. The set of primal optimal solutions is bounded if and only if there exists a nonnegative linear combination of the rows of M which is positive. When M is empty the primal optimal solution is unique.

AMS (MOS) Subject Classification: 90C05

Key Words: Linear programming, simplex method, uniqueness, boundedness

Work Unit Number 5 - Operations Research

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SIGNIFICANCE AND EXPLANATION

Linear programming problems are fundamental to operations research and related areas. The simplex method and its variants are the basic tools for solving these problems. In this report we characterize those linear programming problems that have unique solutions and those that have bounded solutions in terms of information available once the problem is solved by the simplex method.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

OPTIMAL SIMPLEX TABLEAU CHARACTERIZATION
OF UNIQUE AND BOUNDED SOLUTIONS OF LINEAR PROGRAMS

O. L. Mangasarian

1. Introduction

In [4] the author gave a number of equivalent characterizations of the uniqueness of a solution of a general linear programming problem. These characterizations did not include an explicit characterization which could be applied directly to the final optimal simplex tableau for the standard linear programmming problem in order to determine whether the particular optimal solution represented by the tableau is unique or not. Such a characterization, given in Theorem 1 below, follows after some nontrivial algebra from Theorem 2(v) [4]. However a simple direct proof of this characterization is also possible and is given here for the sake of completeness. Theorem 2 characterizes the uniqueness of a dual optimal solution in terms of an optimal simplex tableau also.

In [6] Williams gave characterizations of a bounded solution set of a linear program in terms of the initial data of the linear program. In Theorems 4 and 5 we characterize the boundedness of the primal and dual optimal solution sets respectively in terms of an optimal simplex tableau. As expected the boundedness characterizations impose less stringent conditions than the corresponding uniqueness characterizations. The possible and impossible combinations of uniqueness, boundedness and degeneracy of primal and dual optimal solutions are summarized in Table 1. Examples following Table 1 illustrate all the possible combinations.

We introduce now the standard linear program in canonical form [1]

$$\underset{y \in R^n}{\text{Maximize}}\ z = c^T y \quad \text{subject to} \quad Ay \leqq b,\ y \geqq 0 \tag{1}$$

where c and b are given vectors in R^n and R^m respectively, A is a given real $m \times n$ matrix and the superscript T denotes the transpose. We note immediately that uniqueness of a solution $\bar{y}$ to (1) is equivalent to uniqueness of a solution $(\bar{y},\bar{s})$ in R^{n+m} to the

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equivalent problem with slack variable s in R^m

$$\underset{(y,s)\in R^{n+m}}{\text{Maximize}} \quad z = c^T y \quad \text{subject to} \quad s = -Ay + b,\ y \geqq 0,\ s \geqq 0 \ .$$

Define $x = (y,s)$ and assume that after a finite number of pivots the following standard optimal simplex tableau [1,5] has been obtained after a column and row rearrangement if necessary.

$$\begin{array}{cccc|c|l}
x_{B_+} & x_{B_0} & x_{N_+} & x_{N_0} & = 1 & \\
\hline
I & 0 & L_+ & M_+ & + & x_{B_+} \\
0 & I & L_0 & M_0 & 0 & x_{B_0} \\
\hline
0 & 0 & + & 0 & Q & z \\
\hline
u_{N_+} & u_{N_0} & u_{B_+} & u_{B_0} & &
\end{array} \qquad (3)$$

This is equivalent to the following condensed or Tucker tableau [2,7]

$$\begin{array}{ccc|cc|c}
 & & & -x_{N_+} & -x_{N_0} & 1 \\
\hline
u_{N_+} & x_{B_+} & = & L_+ & M_+ & + \\
u_{N_0} & x_{B_0} & & L_0 & M_0 & 0 \\
\hline
1 & z & = & + & 0 & Q \\
\hline
 & & & u_{B_+} = & u_{B_0} = & w=
\end{array} \qquad (3')$$

For convenience define

$$L = \begin{bmatrix} L_0 & M_0 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} M_+ \\ M_0 \end{bmatrix} \ .$$

In the above tableaus the symbols are defined as follows:

x_{B_+} = primal optimal positive basic variables (with values denoted by + in rightmost column of tableau (3))

x_{B_0} = primal optimal zero basic variables (with values denoted by 0 in rightmost column of tableau (3))

u_{B_+} = dual optimal positive basic variables (with values denoted by + in bottom row of tableau (3))

u_{B_0} = dual optimal zero basic variables (with values denoted by 0 in bottom row of tableau (3))

x_{N_+} = primal optimal (zero) nonbasic variables corresponding to u_{B_+}

x_{N_0} = primal optimal (zero) nonbasic variables corresponding to u_{B_0}

u_{N_+} = dual optimal (zero) nonbasic variables corresponding to x_{B_+}

u_{N_0} = dual optimal (zero) nonbasic variables corresponding to x_{B_0}

I = identity matrix of appropriate dimension

M_+ = matrix in tableau (3) with rows corresponding to x_{B_+} and columns corresponding to u_{B_0}

M_0 = matrix in tableau (3) with rows corresponding to x_{B_0} and columns corresponding to u_{B_0}

L_+ = matrix in tableau (3) with rows corresponding to x_{B_+} and columns corresponding to u_{B_+}

L_0 = matrix in tableau (3) with rows corresponding to x_{B_0} and columns corresponding to u_{B_+}

$x_B = (x_{B_+} \;\; x_{B_0})$

$x_N = (x_{N_+} \;\; x_{N_0})$

$u_B = (u_{B_+} \;\; u_{N_0})$

$u_N = (u_{N_+} \;\; u_{N_0})$

w = dual objective function

Q = maximum value of the primal objective function on the feasible region.

Our principal results, contained in Theorems 1 to 5, are given in terms of the matrices M_0, M and L of the optimal tableau (3) and can be summarized as follows:

(1) <u>Primal uniqueness</u> $x_{B_0} \neq \emptyset$ whenever $u_{B_0} \neq \emptyset$ and $p^T M_0 > 0$ for some $p \geqq 0$

(2) <u>Dual uniqueness</u> $u_{B_0} \neq \emptyset$ whenever $x_{B_0} \neq \emptyset$ and $M_0 q < 0$ for some $q \geqq 0$

(3) <u>Primal and dual uniqueness</u> $x_{B_0} = \emptyset$ and $u_{B_0} = \emptyset$

(4) <u>Primal boundedness</u> $r^T M > 0$ for some $r \geqq 0$

(5) <u>Dual boundedness</u> $Lt < 0$ for some $t \geqq 0$.

2. Uniqueness of Solution

With the aid of the optimal tableau (3) it is possible to characterize the uniqueness of a primal optimal solution as follows.

Theorem 1 (Uniqueness of primal optimal solution). The primal optimal solution to the linear program (1), $x_{B_+} > 0$, $x_{B_0} = 0$, $x_{N_+} = 0$, $x_{N_0} = 0$, is unique if and only if x_{B_0} nonvacuous whenever u_{B_0} is nonvacuous and there exists a $p \geqq 0$ such that $p^T M_0 > 0$.

Proof. The condition that there exists a p such that $p \geqq 0$ and $p^T M_0 > 0$ is equivalent by Motzkin's theorem of the alternative [3] to

$$-M_0 q \geqq 0, \quad 0 \neq q \geqq 0 \quad \text{has no solution } q \quad . \tag{4}$$

We establish now the necessity and sufficiency of condition (4) for the uniqueness of the solution (x_B, x_N).

(Necessity) Let (x_B, x_N) be a unique solution of (1). If u_{B_0} is empty then condition (4) is vacuously satisfied because M_0 is vacuous. Suppose now that u_{B_0} is nonvcuous then x_{B_0} is nonvacuous, else for sufficiently small positive λ and for a vector e of ones the point $\tilde{x}_{B_+} = x_{B_+} - \lambda M_+ e$, $\tilde{x}_{N_+} = 0$, $\tilde{x}_{N_0} = \lambda e$ is primal feasible and distinct from $(x_{B_+}, x_{N_+}, x_{N_0})$ and the corresponding value of the objective function is $\tilde{z} = Q$ contradicting the uniqueness of $(x_{B_+}, x_{N_+}, x_{N_0})$. Hence M_0 is nonvacuous. Suppose now that there exists a q such that $-M_0 q \geqq 0$ and $0 \neq q \geqq 0$, thus violating (4). We will show that this contradicts the uniqueness of (x_B, x_N). For a sufficiently small positive number λ, the point

$$\tilde{x}_{B_+} = x_{B_+} - \lambda M_+ q > 0$$

$$\tilde{x}_{B_0} = -\lambda M_0 q \geqq 0$$

$$\tilde{x}_{N_+} = 0$$

$$\tilde{x}_{N_0} = \lambda q \geqq 0 \quad , \quad \lambda q \neq 0 \quad ,$$

is primal feasible and distinct from $(x_{B_+}, x_{B_0}, x_{N_+}, x_{N_0})$ but the corresponding value of the objective function is $\tilde{z} = Q$ thus contrdicting the uniqueness of $(x_{B_+}, x_{B_0}, x_{N_+}, x_{N_0})$.

(Sufficiency) If u_{B_0} is empty then for any other primal feasible point $(\tilde{x}_B, \tilde{x}_N)$ distinct from (x_B, x_N) at least one component of $\tilde{x}_{N_+}$, say $\left(\tilde{x}_{N_+}\right)_k$ must be positive while $\tilde{x}_N \geqq 0$ in which case the corresponding value of the objective function is

$$\tilde{z} = -u_{B_+}^T \tilde{x}_{N_+} + Q \leqq -\left(u_{B_+}\right)_k \left(\tilde{x}_{N_+}\right)_k + Q < Q$$

and hence $(\tilde{x}_B, \tilde{x}_N)$ cannot be primal optimal and so (x_B, x_N) is unique. Suppose now that u_{B_0} is nonvacuous then x_{B_0} and consequently M_0 are nonvacuous and suppose that (4) holds. We will now show that if (x_B, x_N) is not unique a contradiction ensues. For a distinct optimal solution $(\tilde{x}_B, \tilde{x}_N)$ to exist we need to have $0 \neq (\tilde{x}_{N_+}, \tilde{x}_{N_0}) \geqq 0$. If $0 \neq \tilde{x}_{N_+} \geqq 0$, then $\tilde{z} = -u_{B_+} \tilde{x}_{N_+} + Q < Q$ and hence the point cannot be optimal. So $\tilde{x}_{N_+} = 0$ and $0 \neq \tilde{x}_{N_0} \geqq 0$. Now if for some k-th component of $-M_0 \tilde{x}_{N_0}$, $\left(-M_0 \tilde{x}_{N_0}\right)_k < 0$, it follows that $\left(\tilde{x}_{B_0}\right)_k = \left(-M_0 \tilde{x}_{N_0}\right)_k < 0$ making the point infeasible. Hence $-M_0 \tilde{x}_{N_0} \geqq 0$ and $0 \neq \tilde{x}_{N_0} \geqq 0$, which contradicts (4).

□

Remark 1. In [1,p.95] Dantzig established the sufficiency of the emptiness of u_{B_0} for the uniqueness of the primal optimal solution. This is a special case of Theorem 1 above.

Uniqueness of a solution to the dual linear program

$$\underset{v \in R^m}{\text{Minimize}}\ w = b^T v \quad \text{subject to} \quad A^T v \geqq c,\ v \geqq 0 \tag{5}$$

associated with the linear program (1) can also be obtained by means of the optimal tableau (3). We again note that uniqueness of a solution $\bar{v}$ to (5) is equivalent to uniqueness of a solution $(\bar{v}, \bar{t})$ in R^{m+n} to the equivalent linear program with slack variable t in R^n

$$\underset{(v,t) \in R^{m+n}}{\text{Minimize}}\ w = b^T v \quad \text{subject to} \quad t = A^T v - c,\ v \geqq 0,\ t \geqq 0\ . \tag{6}$$

The combined dual variables v and t are defined as $u = (v,t)$ and appear in tableau (3). By casting (5) into the equivalent format of problem (1)

$$\underset{v \in R^m}{\text{Maximize}} \ -w = -b^T v \quad \text{subject to} \quad -A^T v \geqq -c, \ v \geqq 0 \tag{7}$$

we can characterize uniqueness of its solution by means of tableau (3) as follows.

Theorem 2. (Uniqueness of dual optimal solution) The dual optimal solution to the linear program (1), $u_{B_+} > 0$, $u_{B_0} = 0$, $u_{N_+} = 0$, $u_{N_0} = 0$, is unique if and only if u_{B_0} is nonvacuous whenever x_{B_0} is nonvacuous and there exists a $q \geq 0$ such that $M_0 q < 0$.

By combining Theorems 1 and 2 we can characterize the simultaneous uniqueness of both primal and dual optimal solutions as follows.

Theorem 3. (Uniqueness of primal and dual optimal solutions) The primal and dual optimal solutions to the linear program (1) are both unique if and only if both are nondegenerate, that is x_{B_0} is empty and u_{B_0} is empty.

Proof. If both the primal and dual optimal solutions are nondegenerate then the dual optimal solution is unique by Theorem 2 and the primal optimal solution is unique by Theorem 1. Suppose now that both primal and dual optimal solutions are unique and that one of them is degenerate. We will exhibit a contradiction. If the primal (dual) optimal solution is degenerate then by Theorem 2 (Theorem 1) the dual (primal) optimal solution is also degenerate. Hence both primal and dual optimal solutions are degenerate. By Theorem 1 then there exists a $p \geq 0$ such that $p^T M_0 > 0$ and by Theorem 2 there exists a $q \geq 0$ such that $M_0 q < 0$. Since both p and q are nonzero this then leads to the contradiction

$$0 < (p^T M_0) q = p^T (M_0 q) < 0 \quad .$$

□

3. Boundedness of Solution

Again with the aid of the optimal tableau (3) it is possible to characterize the boundedness of a primal optimal solution set as follows.

Theorem 4. (Boundedness of the primal optimal solution set) The primal optimal solution set to the linear program (1) is bounded if and only if for some or all optimal simplex tableaus such as (3) there exists an $r \geqq 0$ such that $r^T M > 0$.

Proof. Again as in the proof of Theorem 1, the condition that there exists an r such that $r \geqq 0$ and $r^T M > 0$ is equivalent by Motzkin's theorem of the alternative [3] to

$$-Ms \geqq 0,\ 0 \neq s \geqq 0 \text{ has no solution } s \quad .$$

We establish now the necessity and sufficiency of condition (8) for the boundedness of the solution set of (1).

(Necessity) Let (3) be some optimal tableau for problem (1) and let there exist a nonzero s satisfying $s \geqq 0$ and $-Ms \geqq 0$. We will show that this implies that the primal optimal solution set is unbounded. For any positive λ the point

$$\tilde{x}_B = \begin{pmatrix} x_{B_+} \\ x_{B_0} \end{pmatrix} - \lambda Ms \geqq 0$$

$$\tilde{x}_{N_+} = 0$$

$$\tilde{x}_{N_0} = \lambda s \geqq 0$$

is primal feasible, the corresponding value of the objective function is Q and hence is primal optimal. However $\|x_{N_0}\| = \lambda \|s\|$ is unbounded as $\lambda \to \infty$ because $s \neq 0$. Hence the primal optimal solution set is unbounded.

(Sufficiency) If for some optimal tableau (3) u_{B_0} is empty then by Theorem 1, (x_B, x_N) is a unique solution of problem (1). So suppose now that u_{B_0} is nonempty for all optimal tableaus of problem (1) and let (3) be any such optimal tableau. We will show that if (1) has an unbounded primal optimal solution set then there exists a nonzero s such that $s \geqq 0$ and

$-Ms \geqq 0$. Since the primal optimal solution set is unbounded there exists a sequence of nonnegative optimal vectors $\{x_B^i, x_N^i\}$, $i = 1,2,\ldots,$ such that

$$\lim_{i\to\infty} \|x_B^i - x_B, x_N^i - x_N\| = \infty \quad .$$

From tableau (3), since $x_N = (x_{N_+}, x_{N_0}) = 0$, this is equivalent to

$$\lim_{i\to\infty} \left\| -\begin{pmatrix} L_+ \\ L_0 \end{pmatrix} x_{N_+}^i - \begin{pmatrix} M_+ \\ M_0 \end{pmatrix} x_{N_0}^i, \; x_{N_+}^i, x_{N_0}^i \right\| = \infty \quad .$$

If $0 \neq x_{N_+}^i \geqq 0$ then the corresponding value of the objective function is

$$z^i = -u_{B_+} x_{N_+}^i + Q < Q$$

and hence the point (x_B^i, x_N^i) cannot be primal optimal. So $x_{N_+}^i = 0$, $i = 1,2,\ldots,$ and from (9) it follows that $\lim_{i\to\infty} \|x_{N_0}^i\| = \infty$. But

$$x_B^i = -Mx_{N_0}^i + x_B \geqq 0, \; i = 1,2,\ldots \quad .$$

Hence

$$\frac{-Mx_{N_0}^i}{\|x_{N_0}^i\|} + \frac{x_B}{\|x_{N_0}^i\|} \geqq 0, \; i = 1,2,\ldots \quad .$$

Since $\lim_{i\to\infty} \|x_{N_0}^i\| = \infty$ it follows by the Bolzano-Wieirstrass Theorem that the bounded sequence $\left\{ \dfrac{x_{N_0}^i}{\|x_{N_0}^i\|} \right\}$ has an accumulation point s such that $0 \neq s \geqq 0$ and $-Ms \geqq 0$.

□

By the symmetry between (1) and (5), the following result characterizes the boundedness of the dual optimal variables associated with (1)

Theorem 5. (Boundedness of the dual optimal solution set) The dual optimal solution set to the linear program (1) is bounded if and only if for some or all optimal simplex tableaus such as (3) there exists a $t \geqq 0$ such that $Lt < 0$.

4. Summary of Possible Outcomes and Examples

We now summarize for convenience the possible and impossible combinations of uniqueness, boundedness and degeneracy of primal and dual optimal solutions in Table 1. Examples (1) to (7) illustrating all the possible combinations appearing in Table 1 are given following the table.

Table 1

Primal Optimal Solution

Dual Optimal Solution	$\tilde{B}D$	BD $(\tilde{U})$	UD	$\tilde{B}\tilde{D}$	$B\tilde{D}$ $(\tilde{U})$	$U\tilde{D}$
$\tilde{B}D$	$1^{(5)}$	$1^{(6)}$	$1^{(7)}$	0	0	0
$BD(\tilde{U})$	$1^{(6')}$	$1^{(2)}$	$1^{(1')}$	0	0	0
UD	$1^{(7')}$	$1^{(1)}$	0	$1^{(4)}$	$1^{(1)}$	0
$\tilde{B}\tilde{D}$	0	0	$1^{(4')}$	0	0	0
$B\tilde{D}(\tilde{U})$	0	0	$1^{(1')}$	0	0	0
$U\tilde{D}$	0	0	0	0	0	$1^{(3)}$

B = Bounded D = Degenerate U = Unique

$\tilde{B}$ = Unbounded $\tilde{D}$ = Nondegenerate $\tilde{U}$ = Nonunique

$1^{(i)}$ = Possible combination illustrated by Example (i)

$1^{(i')}$ = Possible combination illustrated by Example (i) with the roles of the primal and dual problems interchanged

0 = Impossible combination.

Example 1 (Primal bounded nonunique degenerate/nondegenerate, dual unique degenerate)

$$\text{Max } x_1+x_2 \text{ s.t. } x_1+x_2 \leq 1,\ x_2 \leq 1,\ x_1 \geq 0,\ x_2 \geq 0$$

The primal optimal solution set is $\{x_1, x_2 \mid x_1+x_2 = 1,\ x_1 \geq 0,\ x_2 \geq 0\}$ contains the degenerate vertex $x_1 = 0,\ x_2 = 1$ and the nondegenerate vertex $x_1 = 1,\ x_2 = 0$ which correspond to the following two optimal tableaus respectively where the slacks x_3 and x_4 have been introduced:

x_1	x_2	x_3	x_4	$= 1$	
1	0	1	-1	0	x_1
0	1	0	1	1	x_2
0	0	1	0	1	z
u_3	u_4	u_1	u_2		

x_1	x_4	x_3	x_2	$= 1$	
1	0	1	1	1	x_1
0	1	0	1	1	x_4
0	0	1	0	1	z
u_3	u_2	u_1	u_4		

From the first tableau we observe that both primal and dual solutions are degenerate, the primal solution is nonunique, because $p \cdot (-1) > 0,\ p \geqq 0$ has no solution. The primal solution set is bounded because $(r_1\ r_2)\begin{pmatrix}-1\\1\end{pmatrix} > 0$ has a nonnegative solution $r_1 = 0,\ r_2 = 1$, and the dual solution is unique because $-1 \cdot q < 0,\ q \geqq 0$ has a solution. From the second tableau we observe that the nondegenerate primal solution is nonunique because it is nondegenerate while the dual solution is degenerate. The primal solution is bounded because $(r_1\ r_2)\begin{pmatrix}1\\1\end{pmatrix} > 0$ has a nonnegative solution $r_1 = 1,\ r_2 = 1$. Furthermore the dual solution is unique because the primal solution is nondegenerate. By interchanging the roles of the primal and dual problems this example can also serve to illustrate the case where the dual optimal solution is bounded, nonunique degenerate/nondegenerate while the primal optimal solution is unique and degenerate.

<u>Example 2</u> (Primal and dual bounded nonunique degenerate)

$$\text{Max } x_1+x_2 \text{ s.t. } x_1+x_2 \leqq 1,\ x_1+x_2 \leqq 1,\ x_1 \geqq 0,\ x_2 \geqq 0 \ .$$

The primal optimal solution set is $\{x_1,x_2 \mid x_1+x_2 = 1,\ x_1 \geqq 0,\ x_2 \geqq 0\}$ contains the two degenerate vertices $x_1 = 0,\ x_2 = 1$ and $x_1 = 1,\ x_2 = 0$ which correspond to the following two optimal tableaus respectively with slacks x_3 and x_4:

x_1	x_4	x_3	x_2	$= 1$	
1	0	1	1	1	x_1
0	1	-1	0	0	x_4
0	0	1	0	1	z
u_3	u_2	u_1	u_4		

x_2	x_4	x_1	x_3	$= 1$	
1	0	1	1	1	x_2
0	1	0	-1	0	x_4
0	0	0	1	1	z
u_4	u_2	u_3	u_1		

From these tableaus we observe that both primal and dual solutions are degenerate. The primal solution is not unique because $p \cdot 0 > 0,\ p \geqq 0$ has no solution, but the primal solution is bounded because $(r_1\ r_2)\begin{pmatrix}1\\0\end{pmatrix} > 0$ has a nonnegative solution $r_1 = 1,\ r_2 = 1$. The dual solution is not unique because $0 \cdot q < 0,\ q \geqq 0$ has no solution, but the dual solution is bounded because $(-1\ 0)\begin{pmatrix}t_1\\t_2\end{pmatrix} < 0$ has a solution $t_1 = 1,\ t_2 = 1$.

Example 3 (Primal and dual unique and nondegenerate)

$$\text{Max } x_1+x_2 \text{ s.t. } x_1 \leqq 1,\ x_2 \leqq 1,\ x_1 \geqq 0,\ x_2 \geqq 0 \quad .$$

The unique primal optimal solution is $x_1 = x_2 = 1$ and the unique dual optimal solution : $u_1 = u_2 = 1$. These solutions correspond to the optimal tableau with slacks x_3 and x_4:

	x_1	x_2	x_3	x_4	$= 1$	
	1	0	1	0	1	x_1
	0	1	0	1	1	x_2
	0	0	1	1	2	
	u_3	u_4	u_1	u_2		

We observe from the tableau that both primal and dual optimal solutions are nondegenerate hence they are both unique.

Example 4 (Primal unbounded nondegenerate, dual unique degenerate)

$$\text{Max } x_2 \text{ s.t. } x_2 \leqq 1,\ x_1, x_2 \geqq 0 \quad .$$

The primal optimal solution set $\{x_1, x_2 \mid x_1 \geqq 0,\ x_2 = 1\}$ contains the nondegenerate vertex $x_1 = 0,\ x_2 = 1$ which corresponds to the following optimal tableau where the slack x_3 has been introduced:

	x_2	x_1	x_3	$= 1$	
	1	0	1	1	x_2
	0	0	1	1	z
	u_3	u_2	u_1		

From the tableau we conclude that the primal solution set is unbounded because $r \cdot 0 > 0$, $r \geqq 0$ has no solution. The degenerate dual solution is unique because the primal solution is nondegenerate.

Example 5 (Primal unbounded degenerate, dual unbounded degenerate)

$$\text{Max } x_2 \text{ s.t. } x_2 \leqq 1,\ x_2 \geqq 1,\ x_1, x_2 \geqq 0 \quad .$$

The primal optimal solution set is $\{x_1, x_2 \mid x_1 \geqq 0,\ x_2 = 1\}$ and the dual solution set is $\{u_1, u_2 \mid u_1 - u_2 = 1,\ u_1, u_2 \geqq 0\}$. The primal degenerate vertex solution $x_1 = 0,\ x_2 = 1$ corresponds to the following optimal tableau with slack variables x_3 and x_4:

x_2	x_4	x_1	x_3	$= 1$	
1	0	0	1	1	x_2
0	1	0	1	0	x_4
0	0	0	1	1	
u_4	u_2	u_3	u_1		

From the tableau we conclude that the primal solution set is unbounded because $(r_1\ r_2)\begin{pmatrix}0\\0\end{pmatrix} > 0$ has no nonnegative solution and that the dual solution is also unbounded because $(0\ 1)\begin{pmatrix}t_1\\t_2\end{pmatrix} < 0$, has no nonnegative solution.

Example 6 (Primal bounded nonunique degenerate, dual unbounded degenerate)

$$\text{Max } x_2 \text{ s.t. } x_2 \leqq 1,\ x_2 \geqq 1,\ x_1 \leqq 1,\ x_1, x_2 \geqq 0 \quad .$$

The primal optimal solution set is $\{x_1, x_2 \mid 0 \leqq x_1 \leqq 1,\ x_2 = 1\}$ and the dual optimal solution set is $\{u_1, u_2, u_3 \mid u_1 - u_2 = 1,\ u_3 = 0,\ u_1, u_2 \geqq 0\}$. A primal degenerate vertex solution is $x_1 = 0,\ x_2 = 1$ which corresponds to the following optimal tableau with slack variables x_3, x_4 and x_5:

x_2	x_4	x_5	x_1	$x_3 = 1$		
1	0	0	0	1	1	x_2
0	1	0	0	1	0	x_4
0	0	1	1	0	1	x_5
0	0	0	0	1	1	z
u_5	u_2	u_3	u_4	u_1		

From the tableau we conclude that the primal optimal solution set is bounded because $(r_1\ r_2\ r_3)\begin{pmatrix}0\\0\\1\end{pmatrix} > 0$ has a nonnegative solution $r_1 = r_2 = 0$, $r_3 = 1$. However the primal solution is nonunique because $p \cdot 0 > 0$ has no nonnegative solution. The dual optimal solution set is unbounded because $(0\ 1)\begin{pmatrix}t_1\\t_2\end{pmatrix} < 0$ has no nonnegative solution.

Example 7 (Primal unique degenerate, dual unbounded degenerate)

$$\text{Max } x_1 \text{ s.t. } x_1 \leqq 1,\ -x_1+x_2 \leqq -1,\ x_1, x_2 \geqq 0 \ .$$

The unique primal optimal solution is the degenerate vertex $x_1 = 1$, $x_2 = 0$ and corresponds to the following optimal tableau with slacks x_3 and x_4:

x_1	x_4	x_3	$x_2 = 1$		
1	0	1	0	1	x_1
0	1	1	1	0	x_4
0	0	1	0	1	z
u_3	u_2	u_1	u_4		

From the tableau we conclude that the primal solution is unique because $p \cdot 1 > 0$ has the solution $p = 1$ whereas the dual optimal solution set is unbounded because $(1\ 1)\begin{pmatrix}t_1\\t_2\end{pmatrix} < 0$ has no nonnegative solution.

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Uniqueness and boundedness of solutions of linear programs are characterized in terms of an optimal simplex tableau. Let M denote the submatrix in an optimal simplex tableau with columns corresponding to degenerate optimal dual basic variables. A primal optimal solution is unique if and only if there exists a nonvacuous nonnegative linear combination of the rows of M corresponding to degenerate optimal primal basic variables which is positive. The set of primal optimal solutions is bounded if and only if there exists a nonnegative linear combination of the rows of M which is positive. When M is empty the primal optimal solution is unique.

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